Műhelytanulmányok
Vállalatgazdaságtan Tanszék

1053 Budapest, Veres Pálné u. 36., 1828 Budapest, Pf. 489
(+36 1) 482-5901, fax: 482-5844, www.uni-corvinus.hu/vallgazd

The optimality of Richter's model of repair and waste disposal

Imre Dobos

10. sz. Műhelytanulmány
HU ISSN 1786-3031

2002. február

Budapesti Közgazdaságtudományi és Államigazgatási Egyetem
Vállalatgazdaságtan Tanszék
Veres Pálné u. 36.
H-1053 Budapest
Hungary

The optimality of Richter´s model of repair and waste disposal

Imre Dobos
Budapest University of Economics and Public Administration
Veres Pálné u. 36
H-1053 Budapest
Hungary

Abstract

The aim of the paper is to investigate the model of Richter (1996). The model examines a reverse logistics problem. The proposed inventory holding policy in this model was predetermined. We will show that the proposed predetermined policy is optimal, as well.

Keywords: EOQ model, Production, Recycling, Waste disposal, Cost minimization

Összefoglalás

Richter (1996) egy két műhelyes hulladékkezelési problémát vizsgált. Az vizsgált probléma egy termék, pl. konténer, keresletét analizálja. A konténerekben alkatrészeket szállítanak egy termelő műhelybe. Ha a konténerek kiürülnek, akkor azokat a műhelyben tárolják. A konténereket a műhelyben gyűjtik és a termelési/javítási periódus végén a javítóműhelybe szállítják, ahol azt megjavítják. Ha a konténerekre nincs szükség a termelő műhelyben, akkor azokat, mint hulladékokat kezelik. Ha nem hulladéknak, hanem javíthatónak tartják a használt konténert, akkor átviszik a javító/szerelő műhelybe további felhasználásra. A hulladékként kiselejtezett konténereket termeléssel pótolják. A konténerek iránti igényről feltételezhető, hogy konstans keresletű. A modellt adott készletezési stratégia mellett Richter megoldotta, de nem vizsgálta a stratégia optimalitását. A dolgozat célja az optimális stratégia felkutatása.

Kulcsszavak: Tételnagyság modell, Termelés, Újrafelhasználás, Hulladékkezelés, Költségminimalizálás

1. Introduction

This model describes the production of new and the repair of used products in a first shop and the employment of the products in a second shop. The used products can either be stored at the second shop and then be bought back at the end of the collection interval $[0,T]$ to the first shop for repair, or be disposed somewhere outside. In the first shop lot sizes of newly manufactured products and of repairable products have to be determined in order to meet the constant demand rate of the second shop. Some of the used products are collected at the second shop according to a not necessarily unique repair rate. The share of the used products not provided for repair is called waste disposal rate. The setup number for manufacturing is n and for repair m.

In this model we will prove that the predetermined policy of Richter [1-6] is optimal, as well. To prove this proposition, it will be introduced a general inventory holding strategy and the optimal ordering policies are looked for.

2. Parameters and functioning of the model

Let us assume that we have a general strategy with feasible manufacturing and repair batch sizes and order levels. Our aim is to select the minimum cost strategy among the all possible strategies.

The decision variables of the model:

- q_i^r the ith repair batch size, $i = 1,2, ..., m$, positive,
- t_i^r time length of the use of the ith repair batch, $i = 1,2, ..., m$, positive,
- q_j^m the jth manufacturing batch size, $j = 1,2, ..., n$, positive,
- t_j^m time length of the use of the jth manufacturing batch, $j = 1,2, ..., n$, positive,
- I_0^1 initial inventory level at the stock 1, nonnegative,
- I_k^1 inventory level before the arrive of the ith batch at the stock 1, order point, $k = 1,2,...,m, m+1, ..., n$, nonnegative,
- I_0^2 initial inventory level at the stock 2, nonnegative,
- I_e^3 ending inventory level at the stock 3, nonnegative.

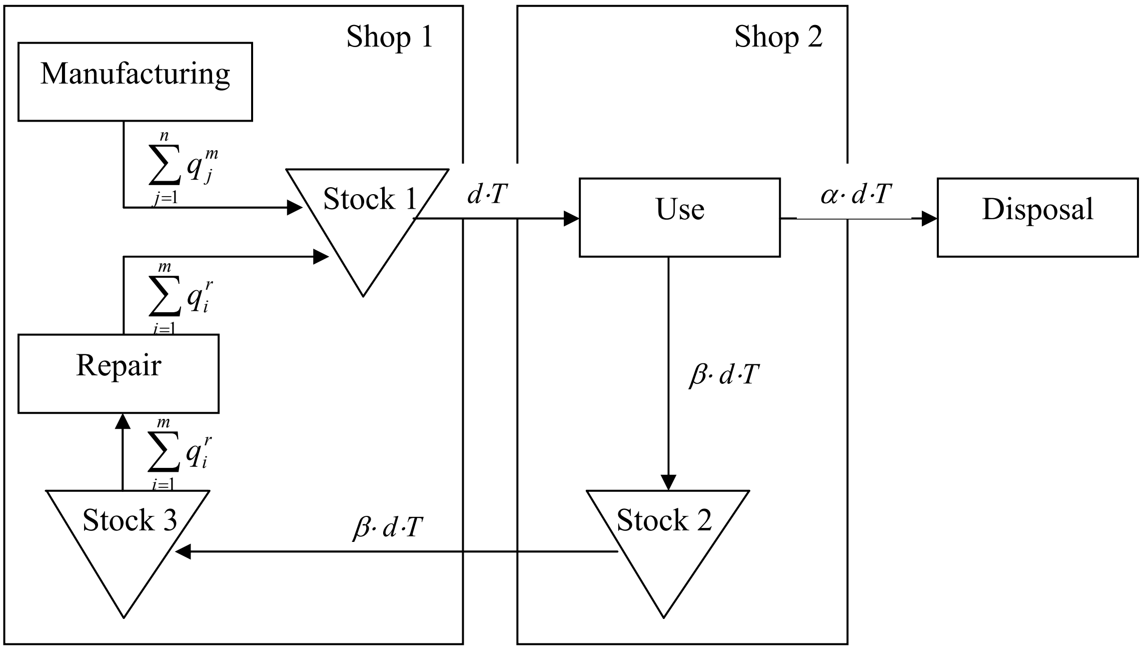
Parameters of the model:

- m repair setup number, $m \geq 1$, integer,
- n production setup number, $n \geq 1$, integer,
- T collection interval,
- d demand rate, units per unit time,
- r fixed repair setup cost,
- s fixed production setup cost,
- h holding cost of serviceable items (stock 1), per unit per time,
- u holding cost of nonserviceable items (stock 2 and 3), per unit per time,
- α the waste disposal rate, percent of the demand rate d, the repair rate is $\beta = 1-\alpha$.

$$\sum_{i=1}^{m} t_i^r + \sum_{i=1}^{n} t_j^m = T$$

$$I_0^3 = I_0^2 + \beta \cdot d \cdot T$$

Figure 1. Material flow of the model



3. Construction of the optimal inventory holding cost function and the order levels

With notations of the previous section we will calculate the inventory holding costs for a general strategy. We want to determine the optimal order levels. The next lemma shows the holding costs for the stock points.

Lemma 1.

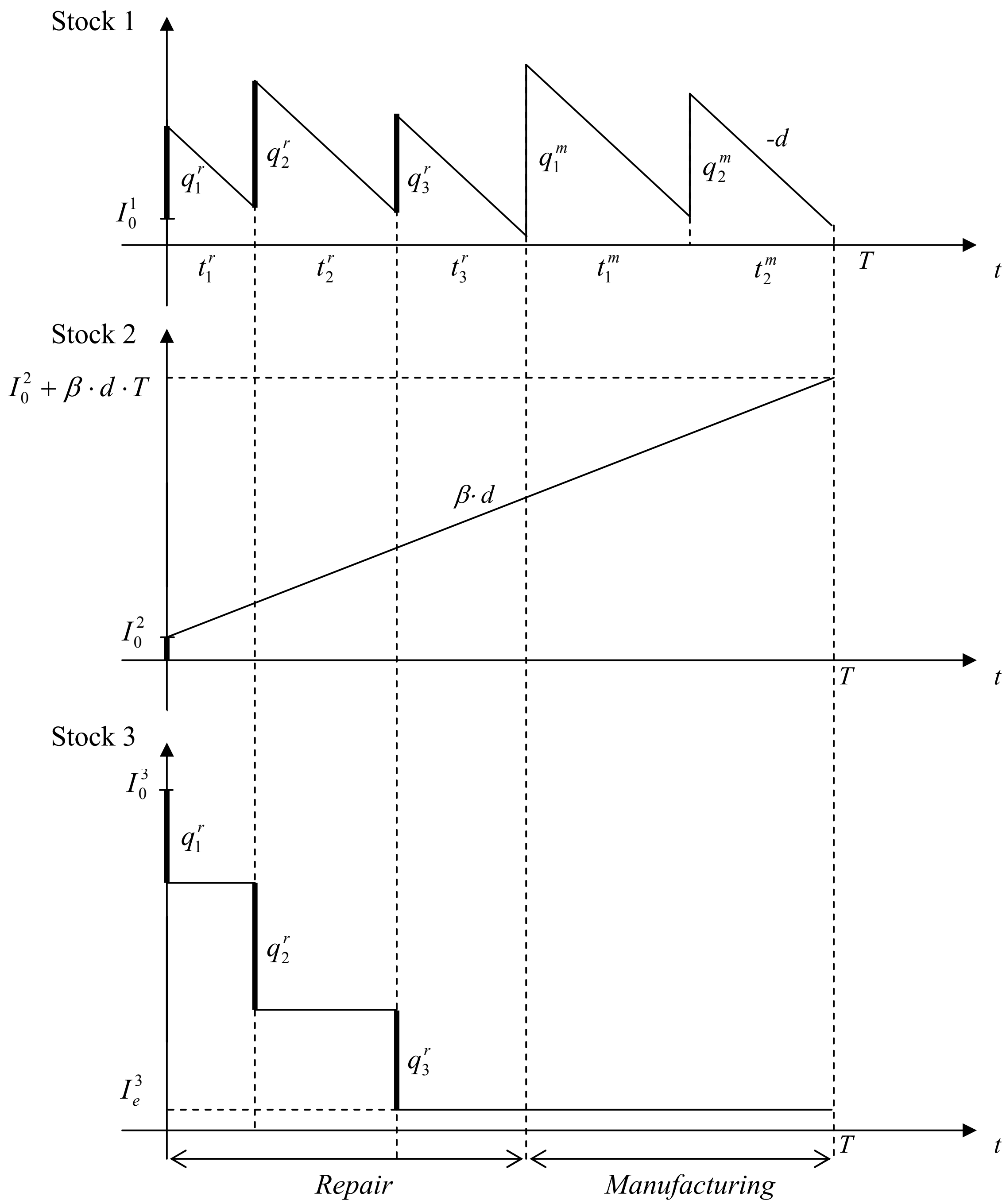
The inventory holding costs H_1, H_2 and H_3 for a strategy $\left\{I_0^1, I_0^2, I_e^3, \left[q_i^r\right]_{i=1}^m, \left[t_i^r\right]_{i=1}^m, \left[q_j^m\right]_{j=1}^n, \left[t_j^m\right]_{j=1}^n\right\}$ in stocks 1,2 and 3 are

$$H_1 = h \cdot \sum_{i=1}^{m} \left(I_i^1 + \frac{d}{2} \cdot t_i^r \right) \cdot t_i^r + h \cdot \sum_{j=1}^{n} \left(I_{m+j}^2 + \frac{d}{2} \cdot t_j^m \right) \cdot t_j^m ,$$

$$H_2 = u \cdot I_0^2 \cdot T + u \cdot \beta \cdot \frac{d}{2} \cdot T^2 ,$$

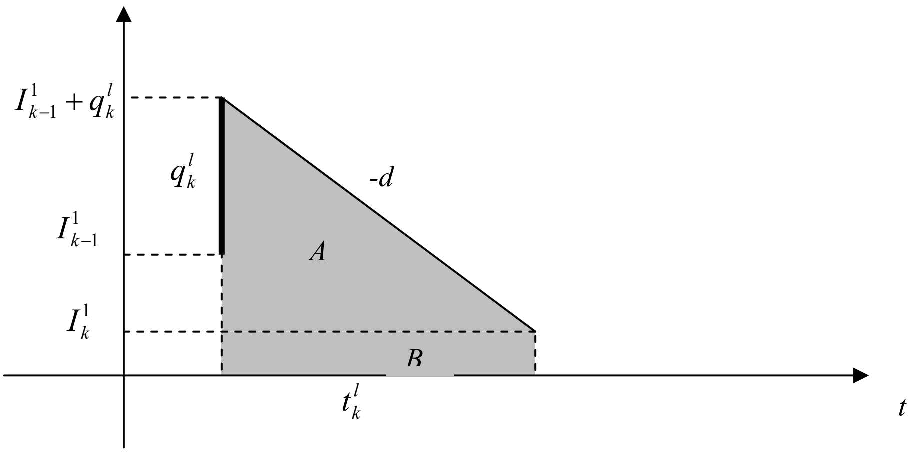
$$H_3 = u \cdot I_e^3 \cdot T + u \cdot \sum_{i=1}^{m-1} t_i^r \cdot \sum_{k=i+1}^{m} q_k^r$$

Figure 2. Inventory levels for a possible strategy at the stock points (*m = 3*, *n = 2*)



Proof. We will prove the first equality, i.e. we calculate the inventory holding costs in the stock 1. The next two equalities can be obtained in a similar way. Let variables q_k^l and t_k^l be the batch size and the use time for the *k*th delivery. If parameter *l* is equal to *r*, then it is a repair batch, in other case, i.e. *l* = *m*, there is a manufacturing batch. Figure 3. illustrates the situation. As we see, the *l*th area consists of the sum of triangle *A* and rectangle *B*. This areas depend on the variables I_k^1 and t_k^l.

Figure 3. Calculation of the holding costs for a repair or manufacturing cycle



Triangle A has an area $\frac{d}{2} \cdot \left(t_k^l\right)^2$. Rectangle B has an area $I_k^1 \cdot t_k^l$. After summation for repair and manufacturing we obtain the holding costs H_1 for stock 1.

We can see that the inventory holding costs for stocks 1 and 2 do not depend on the batch sizes. This property makes easy to determine the optimal ordering policy. The next lemma characterizes the optimal ordering policy.

Lemma 2.

In the optimal inventory holding policy the order levels for stock 1 are equal to zero, the initial inventory level for stock 2 and the ending inventory level for stock 3 are zero, as well:

$$I_k^1 = 0, \quad k = 1, 2, \ldots, m, m+1, \ldots, m+n, \qquad (1)$$

$$I_0^2 = I_e^3 = 0 .$$

Proof. The total inventory olding costs are $H = H_1 + H_2 + H_3$. This can be reformulated in the following form:

$$H = h \cdot \sum_{k=1}^{m+n} I_k^1 \cdot t_k + u \cdot I_0^2 \cdot T + u \cdot I_e^3 \cdot T + h \cdot \frac{d}{2} \cdot \sum_{k=1}^{m+n} \left(t_k\right)^2 + u \cdot \beta \cdot \frac{d}{2} \cdot T^2 + u \cdot \sum_{i=1}^{m-1} t_i^r \cdot \sum_{k=i+1}^{m} q_k^r ,$$

where set $\{t_k\}_{k=1}^{m+n} = \left[\{t_i^r\}_{i=1}^m, \{t_j^m\}_{j=1}^n\right]$ is ordered in the sequence of repair and manufacturing. In this model shortage is not allowed, so the inventory levels are nonnegative. The holding cost function is linear in the inventory levels for all stocks and this fact proves the lemma.

Remark 1. The consequence of the lemma is that for an optimal policy it is necessary to repair or to manufacture the demand of a repair or a manufacturing cycle, i.e. $q_i^r = d \cdot t_i^r, \quad i = 1,2,\ldots,m$ and $q_j^m = d \cdot t_j^m, \quad j = 1,2,\ldots,n$. In this phase of the proof of the optimality it is known that the optimal policy is a saw-tooth for the stock 1. It is not known whether the repair and/or manufacturing order quantities are equal or not.

Remark 2. It follows from the relation (1) that $d\cdot\sum_{i=1}^{m}t_i^r + d\cdot\sum_{i=1}^{n}t_j^m = \sum_{i=1}^{m}q_i^r + \sum_{i=1}^{n}q_j^m = d\cdot T$. The sum of the total batch sizes is equal to the cumulated demand on the collection interval. Because the ending inventory level in stock 2 is equal to the initial inventory level in stock 3 and the optimal ending inventory in stock 3 is zero, it holds $I_0^3 = \sum_{i=1}^{m}q_i^r = \beta\cdot d\cdot T$, i.e. in the optimal policy all not disposed items must be repaired. From this fact it follows that the diposed items must be replaced with manufacturing: $\sum_{i=1}^{n}q_j^m = \alpha\cdot d\cdot T$.

Remark 3. The inventory holding cost function is after this simplification:

$$H = \frac{h}{2\cdot d}\cdot\sum_{i=1}^{m}\left(q_i^r\right)^2 + \frac{h}{2\cdot d}\cdot\sum_{j=1}^{n}\left(q_j^m\right)^2 + \frac{u}{d}\cdot\sum_{i=1}^{m-1}q_i^r\cdot\sum_{k=i+1}^{m}q_k^r + u\cdot\beta\cdot\frac{d}{2}\cdot T^2$$

4. The optimal repair and manufacturing batch sizes

Now the problem has the next form:

$$\frac{h}{2\cdot d}\cdot\sum_{i=1}^{m}\left(q_i^r\right)^2 + \frac{h}{2\cdot d}\cdot\sum_{j=1}^{n}\left(q_j^m\right)^2 + \frac{u}{d}\cdot\sum_{i=1}^{m-1}q_i^r\cdot\sum_{k=i+1}^{m}q_k^r + u\cdot\beta\cdot\frac{d}{2}\cdot T^2 \to min$$

subject to

$$\sum_{i=1}^{m}q_i^r = \beta\cdot d\cdot T,$$

$$\sum_{i=1}^{n}q_j^m = \alpha\cdot d\cdot T,$$

and

$$\begin{aligned} &q_i^r > 0, \quad i = 1, 2, \ldots, m \\ &q_j^m > 0, \quad j = 1, 2, \ldots, n \end{aligned}.$$

The problem in this form is a convex quadratic programming problem, so the necessary conditions of optimality are sufficient, as well. To solve this problem we apply the Lagrange function:

$$L\left(\left[q_i^r\right]_{i=1}^m, \left[q_j^m\right]_{j=1}^n\right) = \frac{h}{2\cdot d}\cdot\sum_{i=1}^{m}\left(q_i^r\right)^2 + \frac{h}{2\cdot d}\cdot\sum_{j=1}^{n}\left(q_j^m\right)^2 + \frac{u}{d}\cdot\sum_{i=1}^{m-1}q_i^r\cdot\sum_{k=i+1}^{m}q_k^r + u\cdot\beta\cdot\frac{d}{2}\cdot T^2 +$$
$$+\lambda_1\cdot\left(\beta\cdot d\cdot T - \sum_{i=1}^{m}q_i^r\right) + \lambda_2\cdot\left(\alpha\cdot d\cdot T - \sum_{j=1}^{n}q_j^m\right)$$

Let us differentiate the Lagrange function in variables q_i^r and q_j^m. Then the necessary and sufficient conditions of optimality are

$$\frac{\partial L}{\partial q_i^r}\left(\left[q_i^r\right]_{i=1}^m,\left[q_j^m\right]_{j=1}^n\right)=\frac{h}{d}\cdot q_i^r+\frac{u}{d}\cdot\sum_{k=1,\,k\neq i}^{m}q_k^r-\lambda_1=0\,,\qquad i=1,2,\ldots,m\,,$$

$$\frac{\partial L}{\partial q_j^m}\left(\left[q_i^r\right]_{i=1}^m,\left[q_j^m\right]_{j=1}^n\right)=\frac{h}{d}\cdot q_j^m-\lambda_2=0\,,\qquad j=1,2,\ldots,n\,.$$

If we calculate the explicit solution for variables q_i^r and q_j^m, then we have the next solution:

$$\frac{d}{h-u}\cdot\lambda_1-\frac{u}{h-u}\cdot\beta\cdot d\cdot T=q_k^r=q^r\,,\quad i=1,2,\ldots,m\,,$$

$$\frac{d}{h}\cdot\lambda_2=q_j^m=q^m\,,\qquad j=1,2,\ldots,n\,.$$

The Langrange multiplier are

$$\lambda_1=\frac{h-u}{d}\cdot\frac{\beta\cdot d\cdot T}{m}\,,\quad i=1,2,\ldots,m\,,$$

$$\lambda_2=\frac{h}{d}\cdot\frac{\alpha\cdot d\cdot T}{n}\,,\qquad j=1,2,\ldots,n\,.$$

With this equations we have solved the problem totally and it is true the

Theorem 1.

The optimal inventory holding policy in the model of Richter with given repair and manufacturing batch numbers is to repair all returned items in equal batch sizes and to manufacture replacing the diposed products in equal batch sizes. The pre-determined policy proposed by Richter is an optimal inventory holding policy with the following batch sizes:

$$q^r=\frac{\beta\cdot d\cdot T}{m}\ \text{and}\ q^m=\frac{\alpha\cdot d\cdot T}{n}\,.$$

Remark 4. In paper of Richter it was introduced the total lot size $x = d\cdot T$. If we reformulate the results of the theorem, then $q^r=\frac{\beta\cdot x}{m}$ and $q^m=\frac{\alpha\cdot x}{n}$, as it was obtained by him. The optimal inventory holding cost function then

$$\frac{h}{2\cdot d}\cdot\frac{\beta^2\cdot x^2}{m}+\frac{h}{2\cdot d}\cdot\frac{\alpha^2\cdot x^2}{n}+\frac{u}{2\cdot d}\cdot\frac{m-1}{m}\cdot\beta^2\cdot x^2+\frac{u}{2\cdot d}\cdot\beta\cdot x^2\,.$$

The total cost function (setup and holding costs) is

$$K_z = (m \cdot r + n \cdot s) + \left(\frac{h}{2 \cdot d} \cdot \frac{\beta^2 \cdot x^2}{m} + \frac{h}{2 \cdot d} \cdot \frac{\alpha^2 \cdot x^2}{n} + \frac{u}{2 \cdot d} \cdot \frac{m-1}{m} \cdot \beta^2 \cdot x^2 + \frac{u}{2 \cdot d} \cdot \beta \cdot x^2 \right).$$

The average cost function is after substitution $T = \frac{x}{d}$

$$K(x, m, n, \alpha) = \frac{K_z}{T} = \frac{d}{x} \cdot (m \cdot r + n \cdot s) + \frac{x}{2} \cdot \left[(h - u) \cdot \frac{\beta^2}{m} + h \cdot \frac{\alpha^2}{n} + u \cdot \beta^2 + u \cdot \beta \right].$$

This model was extensively studied in papers [1-6].

5. The continuous solution of the model

Without mathematical details we give the continuous solution to this model.

Theorem 2. [3, 4]

The optimal lot-size, batch numbers and the cost function in dependence of the disposal rate of this model are

(i) $\alpha \in [0, \alpha_1)$ then

$$x = \sqrt{\frac{2 \cdot d \cdot s}{h \cdot \alpha^2 + u \cdot \beta + u \cdot \beta^2}},$$

$$m(\alpha) = \beta \cdot \sqrt{\frac{s \cdot (h - u)}{r \cdot (h \cdot \alpha^2 + u \cdot \beta + u \cdot \beta^2)}}, \qquad n(\alpha) = 1,$$

$$K(x(\alpha), m(\alpha), n(\alpha), \alpha) = \sqrt{2 \cdot d} \cdot \left[\beta \cdot \sqrt{r \cdot (h - u)} + \sqrt{s \cdot (h \cdot \alpha^2 + u \cdot \beta + u \cdot \beta^2)} \right]$$

(ii) $\alpha \in [\alpha_1, \alpha_2]$ then

$$x = \sqrt{\frac{2 \cdot d \cdot (r + s)}{h \cdot \alpha^2 + h \cdot \beta^2 + u \cdot \beta}},$$

$$m(\alpha) = 1, \qquad n(\alpha) = 1,$$

$$K(x(\alpha), m(\alpha), n(\alpha), \alpha) = \sqrt{2 \cdot d \cdot (r + s) \cdot (h \cdot \alpha^2 + h \cdot \beta^2 + u \cdot \beta)}$$

(iii) $\alpha \in (\alpha_2, 1]$ then

$$x = \sqrt{\frac{2 \cdot d \cdot r}{h \cdot \beta^2 + u \cdot \beta}},$$

$$m(\alpha) = 1, \qquad n(\alpha) = \alpha \cdot \sqrt{\frac{r \cdot h}{s \cdot \left(h \cdot \beta^2 + u \cdot \beta\right)}},$$

$$K(x(\alpha), m(\alpha), n(\alpha), \alpha) = \sqrt{2 \cdot d} \cdot \left[\alpha \cdot \sqrt{s \cdot h} + \sqrt{r \cdot \left(h \cdot \beta^2 + u \cdot \beta\right)}\right]$$

The values α_1 and α_2 are the solution of the next equalities

$$s \cdot (h - u) \cdot (1 - \alpha_1)^2 = r \cdot \left[h \cdot \alpha_1^2 + u \cdot (1 - \alpha_1) + u \cdot (1 - \alpha_1)^2\right]$$

and

$$r \cdot h \cdot \alpha_2^2 = s \cdot \left[h \cdot (1 - \alpha_2)^2 + u \cdot (1 - \alpha_2)\right].$$

It can be shown that $\alpha_1 < \alpha_2$ [3].

6. Conclusion

In this paper th optimality of the predetermined inventory holding policy of Richter was prooved. To prove the optimality, it was introduced a general inventory holding policy and the optimal parameters was determined in a general framework.

In the optimal policy it is effective for lower disposal rate more than one repair batch and only one production/manufacturing batch and for higher disposal rate only one reapir rate and more than one production batch. Between this two regions there is a switching interval where both of the batch numbers are equal to one.

References

[1] Dobos, I., Richter, K. (1999): Comparison of Deterministic One-Product Reverse Logistics Models, in: Hill, R., Smith, D. (Eds.): Inventory Modelling: A Selection of Research Papers Presented at the Fourth ISIR Summer School (1999), Exeter 1999, 69-78

[2] Dobos, I., Richter, K. (2000): The integer EOQ repair and waste disposal model – further analysis. Central European Journal of Operations Research 8, 173-194

[3] Richter, K. (1996): The EOQ repair and waste disposal model with variable setup numbers, European Journal of Operational Research 96, 313-324

[4] Richter, K. (1996): The extended EOQ repair and waste disposal model, International Journal of Production Economics 45, 443-447

[5] Richter, K. (1997): Pure and mixed strategies for the EOQ repair and waste disposal problem, OR Spektrum 19, 123-129

[6] Richter, K. Dobos, I. (1999): Analysis of the EOQ repair and waste disposal model with integer setup numbers, International Journal of Production Economics 59, 463-467